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Generalized Kähler manifolds and off-shell supersymmetry

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Abstract

We solve the long standing problem of finding an off-shell supersymmetric formulation for a general $N = (2, 2)$ nonlinear two dimensional sigma model. Geometrically the problem is equivalent to proving the existence of special coordinates; these correspond to particular superfields that allow for a superspace description. We construct and explain the geometric significance of the generalized Kähler potential for any generalized Kähler manifold; this potential is the superspace Lagrangian.

1 Introduction

Recently general $N = (2, 2)$ supersymmetric sigma models have attracted considerable attention; the renewed interest comes both from physics and mathematics. The physics is related to compactifications with NS-NS fluxes, whereas the mathematics is associated with generalized complex geometry, in particular, generalized Kähler geometry, which is precisely the geometry of the target space of $N = (2, 2)$ supersymmetric sigma models.

The general $N = (2, 2)$ sigma model originally described in [1] has been studied extensively in the physics literature. However, until now an $N = (2, 2)$ off-shell supersymmetric formulation has not been known in the general case. At the physicist's level of rigor, a description in terms $N = (2, 2)$ superfields would imply the existence of a single function that encodes the local geometry–a generalized Kähler potential. Geometrically the problem of $N = (2, 2)$ off-shell supersymmetry amounts to the proper understanding of certain natural local coordinates and the generalized Kähler potential.

In the present work we resolve the issue of what constitutes a complete description of the target space geometry of a general $N = (2, 2)$ sigma model. We show that the full set of fields consists of chiral, twisted chiral and semichiral fields. This was was a natural guess after semichiral superfields were discovered in [2], and was explicitly conjectured by Sevrin and Troost [3]; however, in [4], which contains many useful and interesting results, the erroneous conclusion that this is not the case was reached.

The bulk of the paper is devoted to the proof that certain local coordinates for generalized Kähler geometry exist. From the point of view of $N = (2, 2)$ supersymmetry these coordinates are natural and correspond to the basic superfield ingredients of the model.

The paper is organized as follows. In Section 2 we review the general $N = (2, 2)$ sigma model and describe the generalized Kähler geometry. Section 3 states the problem of off-shell supersymmetry and explains what should be done to solve it. In Section 4 we describe three relevant Poisson structures and their symplectic foliations, and identify coordinates adapted to these foliations. For the sake of clarity in Section 5 we start with a special case when $\ker[J_+, J_-] = \emptyset$. In this case we show that the correct coordinates exist and we explain the existence of the generalized Kähler potential. Next, in Section 6 we extend our results to the general case. Finally, in Section 7 we summarize our results and explain some open problems.

Warning to mathematicians: Due to our background as physicists, we like to work in local coordinates with all indices written out. However, all expressions can be written in coordinate free form, except when we discuss the specific local coordinates in Sections 5 and 6; however, even these local coordinates are merely a convenience, and an appropriate global reformulation of our results certainly exists.

2 Generalized Kähler geometry

In this section we review the results on general $N = (2,2)$ supersymmetric sigma models from the original work [1] (some of these results were found independently in [5, 6]). We define our notation and introduce some relevant concepts.

We start from the general $N = (1,1)$ sigma model written in $N = (1,1)$ superfields (see Appendix A for our conventions)

$$S \propto \int_{\Sigma} d^2\sigma \, d^2\theta \; D_+\Phi^\mu D_-\Phi^\nu (g_{\mu\nu}(\Phi) + B_{\mu\nu}(\Phi)) \; . \tag{2.1}$$

The action (2.1) is manifestly supersymmetric under the usual supersymmetry transformations

$$\delta_1(\epsilon)\Phi^\mu = -i(\epsilon^+ Q_+ + \epsilon^- Q_-)\Phi^\mu \; , \tag{2.2}$$

which form the standard supersymmetry algebra

$$[\delta_1(\epsilon_1), \delta_1(\epsilon_2)]\Phi^\mu = -2i\epsilon_1^+\epsilon_2^+\partial_{\mathbin{+\!\!\!+}}\Phi^\mu - 2i\epsilon_1^-\epsilon_2^-\partial_{=}\Phi^\mu \; . \tag{2.3}$$

We may look for additional supersymmetry transformations of the form [1]

$$\delta_2(\epsilon)\Phi^\mu = \epsilon^+ D_+\Phi^\nu J^\mu_{+\nu}(\Phi) + \epsilon^- D_-\Phi^\nu J^\mu_{-\nu}(\Phi) \; . \tag{2.4}$$

Classically the ansatz (2.4) is unique for dimensional reasons. The action (2.1) is invariant under the transformations (2.4) provided that

$$J^\mu_{\pm\rho} g_{\mu\nu} = -g_{\mu\rho} J^\mu_{\pm\nu} \tag{2.5}$$

and

$$\nabla^{(\pm)}_\rho J^\mu_{\pm\nu} \equiv J^\mu_{\pm\nu,\rho} + \Gamma^{\pm\mu}_{\;\;\rho\sigma} J^\sigma_{\pm\nu} - \Gamma^{\pm\sigma}_{\;\;\rho\nu} J^\mu_{\pm\sigma} = 0 \; , \tag{2.6}$$

where the two affine connections

$$\Gamma^{\pm\mu}_{\;\;\rho\nu} = \Gamma^\mu_{\;\;\rho\nu} \pm g^{\mu\sigma} H_{\sigma\rho\nu} \tag{2.7}$$

have torsion determined by the field strength of $B_{\mu\nu}(\Phi)$:

$$H_{\mu\rho\sigma} = \frac{1}{2}(B_{\mu\rho,\sigma} + B_{\rho\sigma,\mu} + B_{\sigma\mu,\rho}) \; . \tag{2.8}$$

Indeed the functional (2.1) can be rewritten in terms of an extension of H to a ball whose boundary is the surface Σ modulo the usual arguments that apply to the bosonic WZW-term, namely $[H] \in H^3(M, \mathbb{Z})$.

Next we impose the standard on-shell $N = (2,2)$ supersymmetry algebra: The first supersymmetry transformations (2.2) and the second supersymmetry transformations (2.4) automatically commute

$$[\delta_2(\epsilon_1), \delta_1(\epsilon_2)]\Phi^\mu = 0 \ . \tag{2.9}$$

The commutator of two second supersymmetry transformations,

$$\begin{aligned}
[\delta_2(\epsilon_1), \delta_2(\epsilon_2)]\Phi^\mu \ = \ & 2i\epsilon_1^+\epsilon_2^+\partial_{+\!\!+}\Phi^\lambda(J^\mu_{+\nu}J^\nu_{+\lambda}) + 2i\epsilon_1^-\epsilon_2^-\partial_{=}\Phi^\lambda(J^\mu_{-\nu}J^\nu_{-\lambda}) \\
& - \epsilon_1^+\epsilon_2^+ D_+\Phi^\lambda D_+\Phi^\rho \mathcal{N}^\mu_{\lambda\rho}(J_+) - \epsilon_1^-\epsilon_2^- D_-\Phi^\lambda D_-\Phi^\rho \mathcal{N}^\mu_{\lambda\rho}(J_-) \\
& + (\epsilon_1^+\epsilon_2^- + \epsilon_1^-\epsilon_2^+)(J^\mu_{+\nu}J^\nu_{-\lambda} - J^\mu_{-\nu}J^\nu_{+\lambda})(D_+D_-\Phi^\lambda + \Gamma^{-\lambda}_{\ \sigma\nu} D_+\Phi^\sigma D_-\Phi^\nu) \ ,
\end{aligned} \tag{2.10}$$

should satisfy the same algebra as the first (2.3), *i.e.*,

$$[\delta_2(\epsilon_1), \delta_2(\epsilon_2)]\Phi^\mu = -2i\epsilon_1^+\epsilon_2^+\partial_{+\!\!+}\Phi^\mu - 2i\epsilon_1^-\epsilon_2^-\partial_{=}\Phi^\mu \ . \tag{2.11}$$

In (2.10), $\mathcal{N}(J_\pm)$ is the Nijenhuis tensor defined by

$$\mathcal{N}^\rho_{\ \mu\nu}(J) = J^\rho_\lambda \partial_{[\nu} J^\lambda_{\mu]} + \partial_\lambda J^\rho_{[\nu} J^\lambda_{\mu]} \ . \tag{2.12}$$

The field equations that follow from the action (2.1) are

$$D_+D_-\Phi^\mu + \Gamma^{-\mu}_{\ \rho\sigma} D_+\Phi^\rho D_-\Phi^\sigma = 0 \ . \tag{2.13}$$

The first two lines of (2.10) are purely kinematical, *i.e.*, are independent of the form of the action; the last line involves the field equations (2.13), and follows after imposing (2.6).

The algebra (2.10) is the usual supersymmetry algebra (2.3) when $J_\pm$ obey:

$$J^\mu_{\pm\nu} J^\rho_{\pm\mu} = -\delta^\rho{}_\nu \ , \tag{2.14}$$

$$\mathcal{N}^\rho{}_{\mu\nu}(J_\pm) = 0 \ ; \tag{2.15}$$

the last term in (2.10) must also vanish; this is automatic on-shell, *i.e.*, when the field equations (2.13) are satisfied. Thus the on-shell supersymmetry algebra requires that $J_\pm$ are integrable complex structures that preserve the metric; we may introduce the forms $\omega_\pm = gJ_\pm$, which are *not* closed, but satisfy

$$H_{\mu\nu\rho} = \mp J^\lambda_{\pm\mu} J^\sigma_{\pm\nu} J^\gamma_{\pm\rho} (d\omega_\pm)_{\lambda\sigma\gamma} \ , \tag{2.16}$$

as follows from (2.5), (2.6), (2.14) and (2.15).

This is the full description of the most general $N = (2,2)$ sigma model [1]: The target manifold $(M, g, J_\pm, H)$ is a bihermitian complex manifold (*i.e.*, there are two complex

structures and a metric that is Hermitian with respect to both) and the two complex structures must be covariantly constant with respect to connections that differ by the sign of the torsion; this torsion is expressed in terms of a closed 3-form that obeys (2.16).

This bihermitian geometry was first described in [1]. Subsequently, a different geometric interpretation was given in [7], and more recently, following ideas of Hitchin [8], Gualtieri [9] gave an entirely new description of this geometry in terms of generalized complex structures. This geometry is now known as generalized Kähler geometry.

3 $N = (2,2)$ off-shell supersymmetry

In the previous section, the field equations (2.13) are needed to close the supersymmetry algebra. To write the model in a manifestly $N = (2,2)$ covariant form, the algebra must close off-shell. As can be seen from (2.10), the algebra does close off-shell when the two complex structures commute [1]: $[J_+, J_-] = 0$. In this case, both complex structures and the product structure $\Pi = J_+J_-$ are integrable and simultaneously diagonalizable. The manifest $N = (2,2)$ formulation is given in terms of chiral (ϕ) and twisted chiral (χ) scalar superfields:

$$\begin{aligned} &\bar{\mathbb{D}}_\pm \phi = \mathbb{D}_\pm \bar{\phi} = 0 \\ &\bar{\mathbb{D}}_+ \chi = \mathbb{D}_- \chi = \mathbb{D}_+ \bar{\chi} = \bar{\mathbb{D}}_- \bar{\chi} = 0 \ , \end{aligned} \tag{3.17}$$

where $\mathbb{D}$ is the $N = (2,2)$ covariant derivative. The $N = (2,2)$ Lagrangian is a general real function $K(\phi, \bar{\phi}, \chi, \bar{\chi})$, defined modulo (the equivalent of) a Kähler gauge transformation: $f(\phi,\chi) + \bar{f}(\bar{\phi},\bar{\chi}) + g(\phi,\bar{\chi}) + \bar{g}(\bar{\phi},\chi)$. This K serves as a potential both for the metric and for the antisymmetric B-field.

When $[J_+, J_-] \neq 0$, additional (auxiliary) spinorial $N = (1,1)$ fields are needed to close the algebra [10], [11]. The semichiral $N = (2,2)$ scalar superfields introduced in [2] give rise to such auxiliary fields when they are reduced to $N = (1,1)$ superspace. A complex left semichiral superfield $\mathbb{X}_L$ obeys

$$\bar{\mathbb{D}}_+ \mathbb{X}_L = \mathbb{D}_+ \bar{\mathbb{X}}_L = 0 \ , \tag{3.18}$$

and a right semichiral superfield $\mathbb{X}_R$ obeys

$$\bar{\mathbb{D}}_- \mathbb{X}_R = \mathbb{D}_- \bar{\mathbb{X}}_R = 0 \ . \tag{3.19}$$

For these multiplets, the $N = (2,2)$ nonlinear sigma model Lagrangian[1] is the real function $K(\mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R)$, defined modulo $f(\mathbb{X}_L) + \bar{f}(\bar{\mathbb{X}}_L) + g(\mathbb{X}_R) + \bar{g}(\bar{\mathbb{X}}_R)$. Again, the function

[1] In [2], for simplicity, no chiral or twisted chiral multiplets are considered, and hence $[J_+, J_-]$ is invertible.

K is a potential for the metric and the antisymmetric B-field [2]. The target space has generalized Kähler geometry with $[J_+, J_-] \neq 0$ [12]. However, before our work, it was not known if all generalized Kähler geometries with $[J_+, J_-] \neq 0$ admit a description in terms of semichiral superfields.

In [13], it is shown that the kernel of $[J_+, J_-]$ is parametrized completely by chiral and twisted chiral fields. This does not answer the question of whether semichiral multiplets similarily give a complete description of the cokernel. The issue has been addressed, *e.g.*, in [14], [3] and [15].

The general sigma model Lagrangian containing chiral, twisted chiral, and semichiral fields is a real function

$$K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R) \tag{3.20}$$

defined modulo $f(\phi, \chi, \mathbb{X}_L) + g(\phi, \bar{\chi}, \mathbb{X}_R) + \bar{f}(\bar{\phi}, \bar{\chi}, \bar{\mathbb{X}}_L) + \bar{g}(\bar{\phi}, \chi, \bar{\mathbb{X}}_R)$. When there are several multiplets of each kind[2], the fields carry indices

$$\begin{array}{llll} \phi^{\alpha}, \bar{\phi}^{\bar{\alpha}} \ , & \alpha = 1 \ldots d_c \quad , & \chi^{\alpha'}, \bar{\chi}^{\bar{\alpha}'} \ , & \alpha' = 1 \ldots d_t \ , \\ \mathbb{X}^a_L, \bar{\mathbb{X}}^{\bar{a}}_L \ , & a = 1 \ldots d_s \quad , & \mathbb{X}^{a'}_R, \bar{\mathbb{X}}^{\bar{a}'}_R \ , & a' = 1 \ldots d_s \ . \end{array} \tag{3.21}$$

We will also use the collective notation $\mathcal{A} := \{\alpha, \bar{\alpha}\}$, $\mathcal{A}' := \{\alpha', \bar{\alpha}'\}$, $A := \{a, \bar{a}\}$ and $A' := \{a', \bar{a}'\}$. To reduce the $N = (2, 2)$ action to its $N = (1, 1)$ form, we introduce the $N = (1, 1)$ covariant derivatives D and extra supercharges Q:

$$\begin{aligned} D_{\pm} &= \mathbb{D}_{\pm} + \bar{\mathbb{D}}_{\pm} \\ Q_{\pm} &= i(\mathbb{D}_{\pm} - \bar{\mathbb{D}}_{\pm}) \ . \end{aligned} \tag{3.22}$$

In terms of these, the (anti)chiral, twisted (anti)chiral and semi (anti)chiral superfields satisfy

$$\begin{aligned} Q_{\pm}\phi = J_c D_{\pm}\phi \quad &, \quad Q_{\pm}\chi = \pm J_t D_{\pm}\chi \ , \\ Q_{+}\mathbb{X}_L = J_s D_{+}\mathbb{X}_L \quad &, \quad Q_{-}\mathbb{X}_R = J_s D_{-}\mathbb{X}_R \ , \end{aligned} \tag{3.23}$$

where the collective notation is used in the matrices, and where J_c, J_t, and J_s are $2d_c, 2d_t$, and $2d_s$ dimensional canonical complex structures of the form

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \ . \tag{3.24}$$

[2] To be able to integrate out the auxiliary $N = (1, 1)$ spinor superfields, we require an equal number of left and right semichiral superfields $\mathbb{X}_L$ and $\mathbb{X}_R$.

For the pair (ϕ, χ) we use the same letters to denote the $N = (1,1)$ superfields, *i.e.*, the lowest components of the $N = (2,2)$ superfields (ϕ, χ). Each of the semi (anti)chiral fields gives rise to two $N = (1,1)$ fields:

$$\begin{aligned} X_L &\equiv \mathbb{X}_L| \qquad \Psi_{L-} \equiv Q_-\mathbb{X}_L| \\ X_R &\equiv \mathbb{X}_R| \qquad \Psi_{R+} \equiv Q_+\mathbb{X}_R| \ , \end{aligned} \tag{3.25}$$

where a vertical bar means that we take the $\theta^2 \propto \theta - \bar{\theta}$ independent component. The conditions (3.23) then also imply

$$\begin{aligned} Q_+\Psi_{L-} &= J_s D_+ \Psi_{L-} \ , \qquad Q_-\Psi_{L-} = -i\partial_= X_L \\ Q_-\Psi_{R+} &= J_s D_- \Psi_{R+} \ , \qquad Q_+\Psi_{R+} = -i\partial_{+\!\!\!+} X_R \ . \end{aligned} \tag{3.26}$$

Using the relations (3.22)-(3.26) we reduce the $N = (2,2)$ action to its $N = (1,1)$ form according to:

$$\int d^2\xi d^2\theta d^2\bar{\theta}\ K(\phi^{\mathcal{A}}, \chi^{\mathcal{A}'}, \mathbb{X}_L^A, \mathbb{X}_R^{A'})| = \int d^2\xi \mathbb{D}^2\bar{\mathbb{D}}^2 K| = -\frac{i}{4}\int d^2\xi D^2 Q_+ Q_- K| \ . \tag{3.27}$$

Provided that the matrix

$$K_{LR} \equiv \begin{pmatrix} K_{ab'} & K_{a\bar{b}'} \\ K_{\bar{a}b'} & K_{\bar{a}\bar{b}'} \end{pmatrix} \ . \tag{3.28}$$

is invertible, the auxiliary spinors Ψ_{L-}, Ψ_{R+} may be integrated out leaving us with a $N = (1,1)$ second order sigma model action of the type originally discussed in [1]. In (3.28) we use the following notation $K_{ab} \equiv \partial_a \partial_b K$ etc. From this the metric and anti-symmetric B-field may be read off in terms of derivatives of K, and from the form of the second supersymmetry the complex structures $J_\pm$ are determined. In a basis where the coordinates are arranged in a column as

$$\begin{pmatrix} X_L^A \\ X_R^{A'} \\ \phi^{\mathcal{A}} \\ \chi^{\mathcal{A}'} \end{pmatrix} \ , \tag{3.29}$$

and introducing the notation (suppressing the hopefully obvious index structure)

$$\begin{aligned} K_{LR}^{-1} &= (K_{RL})^{-1} \ , \\ C &= JK - KJ = \begin{pmatrix} 0 & 2iK \\ -2iK & 0 \end{pmatrix} , \end{aligned}$$

$$A \;=\; JK+KJ=\begin{pmatrix} 2iK & 0 \\ 0 & -2iK \end{pmatrix}, \tag{3.30}$$

the complex structures read [4]

$$J_+=\begin{pmatrix} J_s & 0 & 0 & 0 \\ K_{RL}^{-1}C_{LL} & K_{RL}^{-1}J_sK_{LR} & K_{RL}^{-1}C_{Lc} & K_{RL}^{-1}C_{Lt} \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & J_t \end{pmatrix} \tag{3.31}$$

and

$$J_-=\begin{pmatrix} K_{LR}^{-1}J_sK_{RL} & K_{LR}^{-1}C_{RR} & K_{LR}^{-1}C_{Rc} & K_{LR}^{-1}A_{Rt} \\ 0 & J_s & 0 & 0 \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & -J_t \end{pmatrix} \tag{3.32}$$

where, *e.g.*, K_{Rc} is the matrix of second derivatives along R- and c-directions, etc. In Sections 5 and 6, where we rederive these expressions from geometrical considerations, we explain the notation in greater detail.

Finally, we compute the $N=(1,1)$ Lagrangian; the sum $E=\frac{1}{2}(g+B)$ of the metric g and B-field takes on the explicit form:

$$\begin{aligned}
E_{LL} &= C_{LL}K_{LR}^{-1}J_sK_{RL} \\
E_{LR} &= J_sK_{LR}J_s+C_{LL}K_{LR}^{-1}C_{RR} \\
E_{Lc} &= K_{Lc}+J_sK_{Lc}J_c+C_{LL}K_{LR}^{-1}C_{Rc} \\
E_{Lt} &= -K_{Lt}-J_sK_{Lt}J_t+C_{LL}K_{LR}^{-1}A_{Rt} \\
E_{RL} &= -K_{RL}J_sK_{LR}^{-1}J_sK_{RL} \\
E_{RR} &= -K_{RL}J_sK_{LR}^{-1}C_{RR} \\
E_{Rc} &= K_{Rc}-K_{RL}J_sK_{LR}^{-1}C_{Rc} \\
E_{Rt} &= -K_{Rt}-K_{RL}J_sK_{LR}^{-1}A_{Rt} \\
E_{cL} &= C_{cL}K_{LR}^{-1}J_sK_{RL} \\
E_{cR} &= J_cK_{cR}J_s+C_{cL}K_{LR}^{-1}C_{RR} \\
E_{cc} &= K_{cc}+J_cK_{cc}J_c+C_{cL}K_{LR}^{-1}C_{Rc} \\
E_{ct} &= -K_{ct}-J_cK_{ct}J_t+C_{cL}K_{LR}^{-1}A_{Rt} \\
E_{tL} &= C_{tL}K_{LR}^{-1}J_sK_{RL} \\
E_{tR} &= J_tK_{tR}J_s+C_{tL}K_{LR}^{-1}C_{RR} \\
E_{tc} &= K_{tc}+J_tK_{tc}J_c+C_{tL}K_{LR}^{1}C_{Rc} \\
E_{tt} &= -K_{tt}-J_tK_{tt}J_t+C_{tL}K_{LR}^{-1}A_{Rt}
\end{aligned} \tag{3.33}$$

It is interesting that there are no corrections from chiral and twisted chiral fields in the semichiral sector (where the results agree with [2] and [12]), whereas in the chiral and twisted chiral sector the semichiral fields contribute substantially.

Thus locally all objects ($J_\pm$, g, B) are given in terms of second derivatives of a single real function K. By construction, the present geometry is generalized Kähler geometry and therefore satisfies all the relations from the previous section. In the rest of the paper we show that (locally) any generalized Kähler manifold has such a description.

4 Poisson structures

In this section we describe three Poisson structures that arise in generalized Kähler geometry. We study these Poisson structures as we will use local coordinates adapted to their foliations. Since the Poisson geometry is rather a novel subject to some physicists, we collect some basic facts in Appendix C.

We start with the two real Poisson structures

$$\pi_\pm \equiv (J_+ \pm J_-)g^{-1} = -g^{-1}(J_+ \pm J_-)^t \ , \tag{4.34}$$

which were introduced in [7] and later rederived by Gualtieri [9]. We can choose local coordinates in a neighborhood of a regular point x_0 of π_- such that[3]

$$\pi_-^{\mathcal{A}\mu} = 0 \ , \tag{4.35}$$

where $\mathcal{A}$ label the coordinates along the kernel of π_-; using (4.34), in these coordinates the complex structures obey

$$J_{+\nu}^{\mathcal{A}} = J_{-\nu}^{\mathcal{A}} \ . \tag{4.36}$$

Repeating the same argument for π_+ we get

$$J_{+\nu}^{\mathcal{A}'} = -J_{-\nu}^{\mathcal{A}'} \ , \tag{4.37}$$

where $\mathcal{A}'$ label the coordinates along the kernel of π_+. Moreover, as the combinations $(\pi_+ \pm \pi_-) \propto J_\pm$ are nondegenerate, the Poisson brackets defined by π_+ and π_- cannot have common Casimir functions[4] which parametrize the kernels of $\pi_\pm$. This means that the directions $\mathcal{A}$ and $\mathcal{A}'$ do not intersect and we can choose coordinates where both the relations (4.36) and (4.37) hold [7]. We denote the remaining directions by A and A'

[3] A regular point x_0 of a Poisson structure π is a point where the rank of π does not vary in a neighborhod of x_0; see Appendix C.

[4] Casimir functions give the coordinates along which a Poisson structure is degenerate; see Appendix C.

(for the moment, we do not distinguish A and A'). Thus we have shown that there exist coordinates, labeled by $\mu = (A, A', \mathcal{A}, \mathcal{A}')$, where

$$J_+ = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & J_t \end{pmatrix} , \qquad J_- = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & -J_t \end{pmatrix} , \tag{4.38}$$

and where J_c, J_t are canonical complex structures defined as in (3.24). The existence of these coordinates was originally shown in [13]. Using Poisson geometry this result is rederived in [7]. We can thus choose local coordinates adapted to the following decomposition

$$\ker(J_+ - J_-) \oplus \ker(J_+ + J_-) \oplus \operatorname{coker}[J_+, J_-] , \tag{4.39}$$

where we use the property

$$[J_+, J_-] = (J_+ - J_-)(J_+ + J_-) = -(J_+ + J_-)(J_+ - J_-) . \tag{4.40}$$

Another important Poisson structure

$$\sigma = [J_+, J_-] g^{-1} \tag{4.41}$$

was introduced in [16]. It is related to the real Poisson structures (4.34):

$$\sigma = \pm(J_+ \mp J_-)\pi_\pm = \mp(J_+ \pm J_-)\pi_\mp . \tag{4.42}$$

The identity (4.40) implies a relation between the kernels of the three structures

$$\ker \sigma = \ker \pi_+ \oplus \ker \pi_- . \tag{4.43}$$

The symplectic leaf for σ is $\operatorname{coker}[J_+, J_-]$. The Poisson structure σ satisfies $J_\pm \sigma J_\pm^t = -\sigma$; this implies that in complex coordinates with respect to either $J_\pm$,

$$\sigma = \sigma^{(2,0)} + \bar{\sigma}^{(0,2)} , \tag{4.44}$$

which implies that the real dimension of the symplectic leaves for σ is a multiple of 4 (this was first proven in [3]). Indeed, σ can be interpreted as the $(2,0) + (0,2)$ projection of *e.g.*, π_+, with respect to either $J = J_\pm$:

$$(1 \pm iJ)\sigma(1 \pm iJ)^t = \mp 2i(1 \pm iJ)\pi_+(1 \pm iJ)^t . \tag{4.45}$$

It turns out that $\sigma^{(2,0)}$ is actually a holomorphic Poisson structure [16]:

$$\bar{\partial}\sigma^{(2,0)} = 0 , \tag{4.46}$$

As discussed above (4.39), we have established that along the kernel of σ, complex coordinates can be simultaneously chosen for both J_+ and J_-. Using the properties of the cokernel of σ, in particular (4.44,4.46), in the next two sections we find natural coordinates along the symplectic leaf of σ as well.

5 Structure of coker$[J_+, J_-]$

To simplify the argument, we first consider the special case when $\ker[J_+, J_-] = \emptyset$ on M and σ is thus invertible; this implies $d_c = d_t = 0$, and the complex dimension of M is $2d_s$.

Since σ is a Poisson structure, the two-form[5]

$$\Omega = \sigma^{-1} \ , \tag{5.47}$$

is closed $d\Omega = 0$; it also satisfies $J_\pm^t \Omega J_\pm = -\Omega$. Choosing complex coordinates with respect to J_+,

$$J_+ = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \equiv \begin{pmatrix} J_s & 0 \\ 0 & J_s \end{pmatrix} , \tag{5.48}$$

we decompose the symplectic form Ω into its $(2,0)$ and $(0,2)$ parts [4]

$$\Omega = \Omega_+^{(2,0)} + \bar{\Omega}_+^{(0,2)} \ . \tag{5.49}$$

Then $d\Omega = 0$ implies

$$\partial \Omega_+^{(2,0)} = 0 \ , \qquad \bar{\partial} \Omega_+^{(2,0)} = 0 \ , \tag{5.50}$$

and its complex conjugate expressions with ∂ ($\bar{\partial}$) being a holomorphic (antiholomorphic) differential. Thus $\Omega_+^{(2,0)}$ is a holomorphic symplectic structure and according to Darboux's theorem we can choose coordinates $\{q^a, \bar{q}^{\bar{a}}, p^a, \bar{p}^{\bar{a}}\}$, $a = 1 \ldots d_s$ such that

$$\Omega_+^{(2,0)} = dq^a \wedge dp^a \ , \qquad \bar{\Omega}_+^{(0,2)} = d\bar{q}^{\bar{a}} \wedge d\bar{p}^{\bar{a}} \ . \tag{5.51}$$

These coordinates are compatible with (5.48); the choice of which coordinates we call q and which we call p is called a polarization.

Alternatively, we can choose complex coordinates with respect to J_-; then we have $\Omega = \Omega_-^{(2,0)} + \bar{\Omega}_-^{(0,2)}$, and $\Omega_-^{(2,0)}$ is again a holomorphic symplectic structure. Thus we can introduce the coordinates $\{Q^{a'}, \bar{Q}^{\bar{a}'}, P^{a'}, \bar{P}^{\bar{a}'}\}$ $a' = 1 \ldots d_s$ such that

$$\Omega_-^{(2,0)} = dQ^{a'} \wedge dP^{a'} \ , \qquad \bar{\Omega}_-^{(0,2)} = d\bar{Q}^{\bar{a}'} \wedge d\bar{P}^{\bar{a}'} \ . \tag{5.52}$$

In these coordinates J_- has the form

$$J_- = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \equiv \begin{pmatrix} J_s & 0 \\ 0 & J_s \end{pmatrix} \ . \tag{5.53}$$

[5] This two-form was introduced in [4]; however the authors erroneously concluded that there exist obstructions to the existence of the coordinates that make Ω constant.

The coordinate transformation $\{q,p\} \to \{Q,P\}$ preserves Ω, and hence is a canonical transformations (symplectomorphisms). A canonical transformation can always be described by a generating function K that depends a d_s-dimensional subset of the "old" coordinates $\{q,p\}$ and a d_s-dimensional subset of the "new" coordinates $\{Q,P\}$ (see, *e.g.*, [17]). For simplicity, we choose our polarization such that the generating function K depends on the "old" q and the "new" P coordinates; it is a theorem that such a polarization always exists [17].

Thus in a neighborhood, the canonical transformation is given by the generating function $K(q,P)$

$$p = \frac{\partial K}{\partial q} , \qquad Q = \frac{\partial K}{\partial P} . \tag{5.54}$$

We now calculate J_+, J_- and Ω in the "mixed" coordinates $\{q,P\}$. Consider J_+. In $\{q,P\}$ coordinates J_+ is given by

$$J_+ = \left(\frac{\partial(q,p)}{\partial(q,P)}\right)^{-1} \begin{pmatrix} J_s & 0 \\ 0 & J_s \end{pmatrix} \left(\frac{\partial(q,p)}{\partial(q,P)}\right) . \tag{5.55}$$

The transformation matrix is given as

$$\frac{\partial(q,p)}{\partial(q,P)} = \begin{pmatrix} 1 & 0 \\ \frac{\partial p}{\partial q} & \frac{\partial p}{\partial P} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\partial^2 K}{\partial q \partial q} & \frac{\partial^2 K}{\partial P \partial q} \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ K_{LL} & K_{LR} \end{pmatrix} \tag{5.56}$$

where in complex coordinates we have

$$K_{LL} = \begin{pmatrix} K_{ab} & K_{a\bar{b}} \\ K_{\bar{a}b} & K_{\bar{a}\bar{b}} \end{pmatrix} , \qquad K_{LR} = \begin{pmatrix} K_{ab'} & K_{a\bar{b}'} \\ K_{\bar{a}b'} & K_{\bar{a}\bar{b}'} \end{pmatrix} , \tag{5.57}$$

and we have anticipated our identification the generating function $K(q,P)$ with the action $K(\mathbb{X}_L, \mathbb{X}_R)$ by introducing the labels R, L. We find

$$J_+ = \begin{pmatrix} 1 & 0 \\ -K_{RL}^{-1} K_{LL} & K_{RL}^{-1} \end{pmatrix} \begin{pmatrix} J_s & 0 \\ 0 & J_s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ K_{LL} & K_{LR} \end{pmatrix} = \begin{pmatrix} J_s & 0 \\ K_{RL}^{-1} C_{LL} & K_{RL}^{-1} J_s K_{LR} \end{pmatrix} , \tag{5.58}$$

where K_{LR} and C_{LL} are defined in (3.30) in terms of second derivatives of the generating function K. Thus in the coordinates $\{q,P\}$, J_+ is given by (5.58). Identifying the generating function $K(q,P)$ with the action $K(\mathbb{X}_L, \mathbb{X}_R)$, this result coincides with the one we get from the semichiral sigma model [2, 4] (c.f. (3.31) with no chiral or twisted chiral fields.).

Next we calculate J_- in $\{q,P\}$ coordinates

$$J_- = \left(\frac{\partial(Q,P)}{\partial(q,P)}\right)^{-1} \begin{pmatrix} J_s & 0 \\ 0 & J_s \end{pmatrix} \left(\frac{\partial(Q,P)}{\partial(q,P)}\right) , \tag{5.59}$$

where

$$\frac{\partial(Q,P)}{\partial(q,P)} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial P} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 K}{\partial q \partial P} & \frac{\partial^2 K}{\partial P \partial P} \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} K_{RL} & K_{RR} \\ 0 & 1 \end{pmatrix}. \tag{5.60}$$

In complex coordinates $K_{RL} = (K_{LR})^t$ defined as in (5.57) and K_{RR} is

$$K_{RR} = \begin{pmatrix} K_{a'b'} & K_{a'\bar{b}'} \\ K_{\bar{a}'b'} & K_{\bar{a}'\bar{b}'} \end{pmatrix}. \tag{5.61}$$

Thus we can rewrite (5.59) as

$$J_- = \begin{pmatrix} K_{LR}^{-1} & -K_{LR}^{-1}K_{RR} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} J_s & 0 \\ 0 & J_s \end{pmatrix} \begin{pmatrix} K_{RL} & K_{RR} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} K_{LR}^{-1}J_sK_{RL} & K_{LR}^{-1}C_{RR} \\ 0 & J_s \end{pmatrix}, \tag{5.62}$$

where C_{RR} was defined in (3.30). Once more, we have reproduced the semichiral expression (c.f. (3.32)).

Finally Ω in coordinates (q, P) is given by

$$\Omega = \begin{pmatrix} 0 & K_{LR} \\ -K_{RL} & 0 \end{pmatrix}. \tag{5.63}$$

In these coordinates the metric g is given by [4]

$$g = \Omega[J_+, J_-] \tag{5.64}$$

and this is the same as from semichiral considerations.

Thus we have shown that the metric can be expressed in terms of second derivatives of a single potential K. However, unlike the case of standard Kähler geometry, the metric is not linear in the derivatives of K. It is natural to refer to K as a generalized Kähler potential. This potential has the interpretation simultaneously as a superspace Lagrangian and as the generating function of a canonical transformation[6] between the complex coordinates adapted to J_+ and the complex coordinates adapted to J_-.

Furthermore, recalling that we have assumed $\ker[J_+, J_-] = \emptyset$ throughout this section, the form $(\Omega^{(2,0)})^{d_s}$ is nondegenerate and defines a holomorphic volume form. Thus this is a generalized Calabi-Yau manifold [8].

Finally, one may wonder if there actually exist examples where $\ker[J_+, J_-] = \emptyset$. The work of [2] provides a local example in four-dimensions; in arbitrary dimensions, one can consider hyperkähler manifolds:

[6]This situation was found previously for $N = (4, 4)$ hyperkähler sigma models described in projective superspace [18].

Theorem 1 *A generalized Kähler manifold with the anticommutator of J_+ and J_- constant, i.e., $\{J_+, J_-\} = c\mathbb{I}$, is a hyperkähler manifold whenever $|c| < 2$.*

Proof: Using (2.6), the proof is straightforward in local coordinates. Alternatively one can observe that $B = \Omega\{J_+, J_-\}$ [4], and hence the torsion, which is proportional to dB, vanishes. The explicit complex structures of the hyperkähler manifold can be chosen as:

$$I = J_+ \ , \quad J = \frac{1}{\sqrt{1 - \frac{c^2}{4}}} \left(J_- + \frac{c}{2} J_+ \right) \ , \quad K = IJ \ . \tag{5.65}$$

The construction we have presented can be applied to the hyperkähler case with a new generalized Kähler potential. Indeed from the condition $\{J_+, J_-\} = c\mathbb{I}$, we get a partial differential equation for K in the hyperkähler case. In [3] it has been pointed that in four dimensions, for $c = 0$, this is the Monge-Ampère equation.

6 General case

We now turn to the general case with both $\ker([J_+, J_-])$ and $\mathrm{coker}([J_+, J_-])$ notrivial. Essentially, we have to combine the arguments presented in the two previous sections.

We assume that in a neighborhood of x_0, the ranks of $\pi_\pm$ are constant, and as result, the rank of σ is constant. We work in coordinates adapted to the symplectic foliation of σ. Combining the notations from previous sections, we can chose coordinates $\{q, p, z, z'\}$ in which J_+ has the canonical form

$$J_+ = \begin{pmatrix} J_s & 0 & 0 & 0 \\ 0 & J_s & 0 & 0 \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & J_t \end{pmatrix} , \tag{6.66}$$

where we use the notation (3.24). The coordinates z and z' parametrize the kernels of $\pi_\mp$, respectively. Thus $\{z, z'\}$ parametrize the kernel of σ and $\{q, p\}$ are the Darboux coordinates for a symplectic leaf. On a leaf the symplectic form is given by (5.51). Alternatively we can choose the coordinates $\{Q, P, z, z'\}$ in which J_- has a canonical form[7]

$$J_- = \begin{pmatrix} J_s & 0 & 0 & 0 \\ 0 & J_s & 0 & 0 \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & -J_t \end{pmatrix} . \tag{6.67}$$

[7] We chose signs that are consistent with the sigma model results.

Again (Q, P) are the Darboux coordinates on a leaf with the symplectic form given by (5.52). If we fix a leaf (*i.e.*, put (z, z') to a fixed value) then we can apply the discussion from Section 5. Thus we can choose new coordinates $\{q, P\}$) along a leaf in a neighborhood of (q_0, p_0) (see the discussion of the existence of these coordinates in Section 5). There exists a generating function K such that the relations (5.54) are satisfied. This argument can be a applied to a single leaf. If we change to another leaf then we get another generating function. Thus in a neighborhood of x_0 we have a family[8] of generating functions $K(q, P, z, z')$ such that

$$p = \frac{\partial K}{\partial q}\ , \qquad Q = \frac{\partial K}{\partial P} \tag{6.68}$$

is satisfied. With this definition, $K(q, P, z, z')$ is defined up to the addition of an arbitrary function $f(z, z')$.

Now we can calculate $J_\pm$ in the coordinates $\{q, P, z, z'\}$; the complex structure J_+ is

$$J_+ = \left(\frac{\partial(q,p,z,z')}{\partial(q,P,z,z')}\right)^{-1} \begin{pmatrix} J_s & 0 & 0 & 0 \\ 0 & J_s & 0 & 0 \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & J_t \end{pmatrix} \left(\frac{\partial(q,p,z,z')}{\partial(q,P,z,z')}\right). \tag{6.69}$$

The transformation matrix is given as

$$\begin{aligned} \frac{\partial(q,p,z,z')}{\partial(q,P,z,z')} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\partial p}{\partial q} & \frac{\partial p}{\partial P} & \frac{\partial p}{\partial z} & \frac{\partial p}{\partial z'} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\partial^2 K}{\partial q \partial q} & \frac{\partial^2 K}{\partial P \partial q} & \frac{\partial^2 K}{\partial z \partial q} & \frac{\partial^2 K}{\partial z' \partial q} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ K_{LL} & K_{LR} & K_{Lc} & K_{Lt} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{6.70}$$

where in complex coordinates K_{LL} and K_{LR} were defined in (5.57) and

$$K_{Lc} = \begin{pmatrix} K_{a\alpha} & K_{a\bar{\alpha}} \\ K_{\bar{a}\alpha} & K_{\bar{a}\bar{\alpha}} \end{pmatrix}, \qquad K_{Lt} = \begin{pmatrix} K_{a\alpha'} & K_{a\bar{\alpha}'} \\ K_{\bar{a}\alpha'} & K_{\bar{a}\bar{\alpha}'} \end{pmatrix}. \tag{6.71}$$

[8]One may wonder if the dependence of K on z and z' is smooth; this is necessary to write the coordinate transformation to $\{q, P, z, z'\}$. The existence of these coordinates follows from Arnold's result [17].

Next using (6.69) and (6.70) we calculate J_+

$$J_+ = \begin{pmatrix} J_s & 0 & 0 & 0 \\ K_{RL}^{-1}C_{LL} & K_{RL}^{-1}J_sK_{LR} & K_{RL}^{-1}C_{Lc} & K_{RL}^{-1}C_{Lt} \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & J_t \end{pmatrix}, \tag{6.72}$$

where all of the C matrices are defined in (3.30). This is exactly the same expression one gets from the sigma model considerations (3.31).

Similarly, we calculate the form of J_- in $\{q, P, z, z'\}$ coordinates:

$$J_- = \left(\frac{\partial(Q, P, z, z')}{\partial(q, P, z, z')}\right)^{-1} \begin{pmatrix} J_s & 0 & 0 & 0 \\ 0 & J_s & 0 & 0 \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & -J_t \end{pmatrix} \left(\frac{\partial(Q, P, z, z')}{\partial(q, P, z, z')}\right). \tag{6.73}$$

$$J_- = \begin{pmatrix} K_{LR}^{-1}J_sK_{RL} & K_{LR}^{-1}C_{RR} & -K_{LR}^{-1}C_{Rc} & K_{LR}^{-1}A_{Rt} \\ 0 & -J_s & 0 & 0 \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & -J_t \end{pmatrix}, \tag{6.74}$$

where again the C and A matrices were defined in (3.30) and K_{Rc} and K_{Rt} are

$$K_{Rc} = \begin{pmatrix} K_{a'\alpha} & K_{a'\bar{\alpha}} \\ K_{\bar{a}'\alpha} & K_{\bar{a}'\bar{\alpha}} \end{pmatrix}, \qquad K_{Rt} = \begin{pmatrix} K_{a'\alpha'} & K_{a'\bar{\alpha}'} \\ K_{\bar{a}'\alpha'} & K_{\bar{a}'\bar{\alpha}'} \end{pmatrix}. \tag{6.75}$$

This is exactly the same expression one gets from the sigma model (3.32).

We now consider the metric; in the coordinates $\{q, P, z, z'\}$, the metric has a form

$$g = \begin{pmatrix} g_{AB} & g_{AB'} & g_{A\mathcal{B}} & g_{A\mathcal{B}'} \\ g_{A'B} & g_{A'B'} & g_{A'\mathcal{B}} & g_{A'\mathcal{B}'} \\ g_{\mathcal{A}B} & g_{\mathcal{A}B'} & g_{\mathcal{A}\mathcal{B}} & g_{\mathcal{A}\mathcal{B}'} \\ g_{\mathcal{A}'B} & g_{\mathcal{A}'B'} & g_{\mathcal{A}'\mathcal{B}} & g_{\mathcal{A}'\mathcal{B}'} \end{pmatrix}. \tag{6.76}$$

The definition (4.41) of the Poisson structure σ determines all the components of the metric g *except* those along the kernel of σ: $g_{\mathcal{A}\mathcal{B}}, g_{\mathcal{A}\mathcal{B}'}, g_{\mathcal{A}'\mathcal{B}}, g_{\mathcal{A}'\mathcal{B}'}$; this matches the ambiguity in the generating function $K(q, P, z, z')$ noted below (6.68). The remaining components of the metric can be expressed in terms of the second derivatives of $K(q, P, z, z')$ using the relation (2.16):

$$J_{+\mu}^{\lambda} J_{+\nu}^{\sigma} J_{+\rho}^{\gamma} (d\omega_+)_{\lambda\sigma\gamma} = -J_{-\mu}^{\lambda} J_{-\nu}^{\sigma} J_{-\rho}^{\gamma} (d\omega_-)_{\lambda\sigma\gamma} \ . \tag{6.77}$$

This is obvious in the Kähler case ($J_+ = J_-$), and was shown to be true whenever the $[J_+, J_-] = 0$ in [1]. In the general case, we argue as follows: choosing the local coordinates (q, P, z, z') we can plug the complex structures (6.72) and (6.74) into (6.77). After this the relation (6.77) becomes a first order partial differential equation for the metric g. The differential equation contains the derivatives of K. However, we know a solution for g (which is indeed expressible completely in terms of the second derivatives of K): it is precisely the expression derived from the sigma model (see the expression for E in Section 3). Similarly, (2.16) can be used to determine the 2-form B in terms of the second derivatives of K.

Thus we have established the existence of a generalization of the concept of a Kähler potential for generalized Kähler geometry. It is natural to refer to this function as a generalized Kähler potential. Of course, as we found in the previous section, the second derivatives of the generalized Kähler potential appear nonlinearly in the metric.

7 Summary and discussion

We have resolved the long standing problem of finding manifestly off-shell supersymmetric formulation for the general $N = (2, 2)$ sigma model. We have shown that the full set of fields which is necessary for the description of general $N = (2, 2)$ sigma model consists of chiral, twisted chiral, and semichiral fields. At the geometrical level this implies important results about the generalized Kähler geometry, in particular the existence of a generalized Kähler potential. Thus for generalized Kähler manifold all the differential geometry can be locally encoded in a single real function. We have presented a geometrical proof of this which is essentially independent of sigma model considerations. The only assumption we made was the regularity of the Poisson structures $\pi_\pm$ in a given neighborhood; presumably, continuity allows one to relax this assumption in most cases of physical interest. In general, it would be interesting to go beyond this assumption; this would require the full apparatus of Poisson geometry, in particular a study of the transversal Poisson structures around x_0.

It follows that one can now discuss the general $N = (2, 2)$ sigma models entirely within the powerful $N = (2, 2)$ superfield formalism. In particular such problem as finding quotients of generalized Kähler manifolds can be studied in all generality in this formalism. We plan to come back to this elsewhere.

From the mathematical point of view, it would be interesting to systematically study the first order PDE for the metric that arises from the equation (6.77). Taking into account the discussion in Section 5, we seem to have some new tools with which to study hyperkähler manifolds.

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A $N = (1,1)$ supersymmetry

In this and the next appendix we collect our notation for $N = (1,1)$ and $N = (2,2)$ superspace. In our conventions we closely follow [19].

We use real (Majorana) two-component spinors $\psi^\alpha = (\psi^+, \psi^-)$. Spinor indices are raised and lowered with the second-rank antisymmetric symbol $C_{\alpha\beta}$, which defines the spinor inner product:

$$C_{\alpha\beta} = -C_{\beta\alpha} = -C^{\alpha\beta} \ , \qquad C_{+-} = i \ , \qquad \psi_\alpha = \psi^\beta C_{\beta\alpha} \ , \qquad \psi^\alpha = C^{\alpha\beta}\psi_\beta \ . \tag{A.1}$$

Throughout the paper we use $(+\!\!\!+, =)$ as worldsheet indices, and $(+, -)$ as two-dimensional spinor indices. We also use superspace conventions where the pair of spinor coordinates of the two-dimensional superspace are labelled $\theta^\pm$, and the spinor derivatives $D_\pm$ and supersymmetry generators $Q_\pm$ satisfy

$$\begin{aligned} D_+^2 &= i\partial_{+\!\!\!+} \ , \qquad D_-^2 = i\partial_= \ , \qquad \{D_+, D_-\} = 0 \ , \\ Q_\pm &= iD_\pm + 2\theta^\pm \partial_{\substack{+\!\!\!+ \\ =}} \ , \end{aligned} \tag{A.2}$$

where $\partial_{\substack{+\!\!\!+ \\ =}} = \partial_0 \pm \partial_1$. The supersymmetry transformation of a superfield Φ is given by

$$\begin{aligned} \delta\Phi &\equiv -i(\varepsilon^+ Q_+ + \varepsilon^- Q_-)\Phi \\ &= (\varepsilon^+ D_+ + \varepsilon^- D_-)\Phi - 2i(\varepsilon^+\theta^+\partial_{+\!\!\!+} + \varepsilon^-\theta^-\partial_=)\Phi \ . \end{aligned} \tag{A.3}$$

The components of a scalar superfield Φ are defined by projection as follows:

$$\Phi| \equiv X \ , \qquad D_\pm\Phi| \equiv \psi_\pm \ , \qquad D_+D_-\Phi| \equiv F \ , \tag{A.4}$$

where the vertical bar $|$ denotes "the $\theta = 0$ part". The $N = (1,1)$ spinorial measure is conveniently written in terms of spinor derivatives:

$$\int d^2\theta \ \mathcal{L} = (D_+D_-\mathcal{L})\Big| \ . \tag{A.5}$$

B $N = (2,2)$ supersymmetry

In $N = (2,2)$ superspace, we have two independent $N = (1,1)$ subalgebras with spinor derivatives D^1_α, D^2_α; we define complex complex spinor derivatives

$$\mathbb{D}_\alpha \equiv \frac{1}{2}(D^1_\alpha + iD^2_\alpha) \ , \qquad \bar{\mathbb{D}}_\alpha = \frac{1}{2}(D^1_\alpha - iD^2_\alpha) \tag{B.1}$$

which obey the algebra

$$\begin{aligned} \{\mathbb{D}_+, \bar{\mathbb{D}}_+\} &= i\partial_{+\!\!\!+} \ , \qquad \{\mathbb{D}_-, \bar{\mathbb{D}}_-\} = i\partial_{=} \ , \\ \{\mathbb{D}_\alpha, \mathbb{D}_\beta\} &= 0 \ , \qquad \{\bar{\mathbb{D}}_\alpha, \bar{\mathbb{D}}_\beta\} = 0 \ . \end{aligned} \tag{B.2}$$

These can be written in terms of complex spinor coordinates:

$$\mathbb{D}_\pm = \partial_\pm + \frac{i}{2}\bar{\theta}^\pm \partial_{\substack{+\\[-1ex]=}} \ , \qquad \bar{\mathbb{D}}_\pm = \bar{\partial}_\pm + \frac{i}{2}\theta^\pm \partial_{\substack{+\\[-1ex]=}} \ . \tag{B.3}$$

In terms of the covariant derivatives, the supersymmetry generators are

$$\mathbb{Q}_\alpha = i\mathbb{D}_\alpha + \theta^\beta \partial_{\alpha\beta} \ , \qquad \bar{\mathbb{Q}}_\alpha = i\bar{\mathbb{D}}_\alpha + \bar{\theta}^\beta \partial_{\alpha\beta} \ . \tag{B.4}$$

The supersymmetry transformation of a superfield Φ is then defined by

$$\delta\Phi = i(\epsilon^\alpha \mathbb{Q}_\alpha + \bar{\epsilon}^\alpha \bar{\mathbb{Q}}_\alpha)\Phi \ . \tag{B.5}$$

Irreducible representations of $N = (2,2)$ obey constraints that are compatible with the algebra (B.2); for example, a chiral superfield ($\bar{\mathbb{D}}_\pm \Phi = 0$) has components defined via projections as follows

$$\Phi| \equiv X \ , \qquad \mathbb{D}_\pm \Phi| \equiv \psi_\pm \ , \qquad \mathbb{D}_+\mathbb{D}_-\Phi| \equiv F \ , \tag{B.6}$$

and a twisted chiral superfield ($\bar{\mathbb{D}}_+\chi = \mathbb{D}_-\chi = 0$) has components:

$$\chi| \equiv \tilde{X} \ , \qquad \mathbb{D}_{+\!\!\!+}\chi| \equiv \tilde{\psi}_+ \ , \qquad \bar{\mathbb{D}}_-\chi| \equiv \tilde{\psi}_- \ , \qquad \mathbb{D}_+\bar{\mathbb{D}}_-\chi| \equiv \tilde{F} \ , \tag{B.7}$$

The $N = (2,2)$ spinorial measure is conveniently written in terms of spinor derivatives:

$$\int d^2\theta \, d^2\bar{\theta} \ \mathcal{L} = (\mathbb{D}_+\mathbb{D}_-\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\mathcal{L})\Big| . \tag{B.8}$$

C Poisson geometry

A (d-dimensional) manifold M is Poisson if it admits an antisymmetric bivector $\pi \in \wedge^2 TM$ that satisfies the differential condition

$$\pi^{\mu\nu}\partial_\nu \pi^{\rho\sigma} + \pi^{\rho\nu}\partial_\nu \pi^{\sigma\mu} + \pi^{\sigma\nu}\partial_\nu \pi^{\mu\rho} = 0 \ . \tag{C.1}$$

If π is invertible, π^{-1} is a symplectic structure. The bivector π defines the conventional Poisson bracket

$$\{f,g\} \equiv \pi(df,dg) = \pi^{\mu\nu}\partial_\mu f\,\partial_\nu g\ , \qquad f(x), g(x) \in C^\infty(M)\ , \tag{C.2}$$

which is a bilinear map $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$. Because of (C.1), the Poisson bracket (C.2) has the ordinary antisymmetry property and satisfies the standard Leibnitz rule and Jacobi identity.

Next we recall that (locally) a Poisson manifold admits a foliation by symplectic leaves. Let M be a Poisson manifold with the Poisson structure $\pi^{\mu\nu}$, $\mu,\nu = 1,2,...,d$; choose a point x_0 such that in its neighborhood $\mathrm{rank}(\pi) = n$ is constant. Such a point is called regular.[9]

A vector field is locally Hamiltonian if it can be written as the contraction of the bivector π with a closed one-form e (locally $e = df$ for some function f). The Lie bracket of two locally Hamiltonian vector fields is again locally Hamiltonian:

$$\text{for } v^\mu_A \equiv \pi^{\mu\nu}\partial_\nu f_A\ , \quad (\mathcal{L}_{v_A} v_B)^\mu \equiv v^\nu_A \partial_\nu v^\mu_B - v^\nu_B\partial_\nu v^\mu_A = \pi^{\mu\rho}\partial_\rho\left((\partial_\nu f_B)\pi^{\nu\lambda}(\partial_\lambda f_A)\right). \tag{C.3}$$

The maximum number of linearly independent locally Hamiltonian vector fields in the neighborhood of a regular point x_0 is clearly $n = \mathrm{rank}(\pi)$; then Frobenius theorem implies that the vector fields locally generate an integral submanifold S through x_0, and it is always possible to introduce the local coordinates $x^\mu = \{x^A, x^i\}$, $A = 1,\ldots,n$, $i = n+1,...,d$ in the neighborhood of x_0 such that S can be described by $x^i = \mathit{constant}$ and x^A are the coordinates on S. The restriction of the Poisson bracket to the functions on the submanifold S is again a Poisson bracket, and is indeed a *nondegenerate* Poisson structure on S. As a result, in the coordinates $x^\mu = \{x^A, x^i\}$, π has the following form

$$\pi^{\mu\nu} = \begin{pmatrix} \pi^{AB} & 0 \\ 0 & 0 \end{pmatrix}. \tag{C.4}$$

Since $\pi^{AB} \equiv \pi|_S$ is nondegenerate, it is the inverse of a symplectic structure on S, and thus the Poisson manifold is foliated by symplectic leaves. In a generic coordinate system, there is a locally complete set of $d-n$ independent Casimir functions $\{f_i(x)\}$ of π which have vanishing Poisson bracket with any function from $C^\infty(\mathcal{M})$. In these coordinates the symplectic leaves are determined locally by the conditions $f_i(x) = \mathit{constant}$.

For further details on the Poisson geometry the reader may consult the book [20].

[9] In general, a non-regular Poisson manifold has singular points where the rank jumps [20]. We do not discuss these points and their neighborhoods here.

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